SECTION 10.2 (10.1): SEQUENCES

Up to this point in Calculus, we've been primarily concerned with functions of so-called **continuous** variables (typically denoted by x, y, t, or θ) such as $f(x) = x^2$ or $g(t) = \sin(t)$ where the independent variable takes on values over an **interval of real numbers**. In this chapter, we study **sequences** which are functions of **discrete** variables (usually denoted by n, m, or k) which are functions whose domains are (subsets of) the **natural** or **whole** numbers (occasionally we may see need to dip into the **integers**.) As an example, consider the function

$$f(n) = \frac{(-1)^n\, n}{n+1}, \quad n \geq 1.$$

In this chapter, unless noted otherwise, we'll assume n takes on discrete natural number values: $n = 1, 2, 3, \ldots$

This means the range of f is a **list** of real numbers as opposed to an **interval**:

$$\{f(1), f(2), f(3), f(4), \ldots\} = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots\right\}$$

A sequence can be thought of as such a list of real numbers: an infinite ordered list.

SEQUENCE NOTATION: We rarely use the 'usual' function notation to describe sequences.

For example, the sequence $f(n) = \dfrac{(-1)^n\, n}{n+1}$, for $n \geq 1$ is denoted by $a_n = \dfrac{(-1)^n\, n}{n+1}$, $n \geq 1$ or $\left\{\dfrac{(-1)^n\, n}{n+1}\right\}_{n=1}^{\infty}$.

Here, the variable n is often called the **index** of the sequence and the individual numbers in the list of the sequence are often called the **terms**. (Which foreshadows what we'll be doing with sequences in short order . . .)

EXAMPLE 1: Write out the first five terms of the following sequences.

1. $a_n = \dfrac{2n-1}{n^2}$, $n \geq 1$

Ans: $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

2. $b_k = \left(\dfrac{2}{3}\right)^k$, $k \geq 0$.

Ans: $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

3. $\left\{\dfrac{(-1)^m}{m+1}\right\}_{m=0}^{\infty}$

Ans: $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

RECURSIVE DESCRIPTIONS: Suppose a sequence is defined as follows: $a_1 = 4$; $a_{n+1} = -\frac{1}{2}\, a_n$ for $n \geq 1$.

We're told the first term of the sequence, $a_1 = 4$ and then given the recipe $a_{n+1} = -\frac{1}{2}\, a_n$, called the **recursion formula** telling us how to calculate new terms from existing terms:

$n = 1$: $a_{1+1} = -\frac{1}{2}\, a_1$ that is $a_2 = -\frac{1}{2}\, a_1$ so $a_2 = -\frac{1}{2}(4) = -2$.

$n = 2$: $a_{2+1} = -\frac{1}{2}\, a_2$ that is $a_3 = -\frac{1}{2}\, a_2$ so $a_3 = -\frac{1}{2}(-2) = 1$.

$n = 3$: $a_{3+1} = -\frac{1}{2}\, a_3$ that is $a_4 = -\frac{1}{2}\, a_3$ so $a_4 = -\frac{1}{2}(1) = -\frac{1}{2}$.

Proceeding, we find $a_5 = \frac{1}{4}$, $a_6 = -\frac{1}{8}$, and so on.

In words, the way we get the next term in the sequence is that we multiply the term we have by $-\frac{1}{2}$.

EXAMPLE 2: Write out the first five terms of the sequences described below.

1. $b_1 = 2000$, $b_{n+1} = (1.01)b_n$ for $n \geq 1$.

   Ans: 2000, 2020, 2040.2, 2060.602, 2081.20802, ...

2. **FACTORIALS:** $f_0 = 1$, $f_n = n\, f_{n-1}$ for $n \geq 1$.

   Ans: 1, 1, 2, 6, 24, ...

3. $c_0 = -3$; $c_{k+1} = c_k + 2$ for $k \geq 1$.

   Ans: -3, -1, 1, 3, 5, ...

4. **FIBONACCI NUMBERS:** $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, $n \geq 1$.

   Ans: 1, 1, 2, 3, 5 ...

NOTE: The factorial sequence f_n is usually denoted by $n!$ and has the alternate (yet less precise) definition:

$$0! = 1, \qquad n! = n(n-1)(n-2)\ldots(1),\ n \geq 1.$$

IMPORTANT CLASSES OF SEQUENCES: Recall that linear functions ($f(x) = mx + b$) describe situations in which a process has a constant rate of change. Exponential functions ($f(x) = a\, b^x$) describe situations in which a process has a constant **relative** rate of change. Both linear and exponential functions have important applications, so it should be no surprise they arise in sequences.

ARITHMETIC SEQUENCES: Arithmetic sequences are linear functions. To construct an arithmetic sequence, begin with some initial value, a, and then add a fixed amount d to generate successive terms:

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

Recursively we can define an arithmetic sequence as: $a_1 = a$, $a_{n+1} = a_n + d$, $n \geq 1$.

The letter 'd' here stands for **common difference**. To see why, we can solve $a_{n+1} = a_n + d$ for d: $d = a_{n+1} - a_n$.

That is, the difference d is the same between successive terms.

It doesn't take too much to see that an explicit formula for a_n is: $a_n = a + (n-1)d$ for $n \geq 1$.

Rewriting, we get $a_n = d\,n + (a - d)$ which shows these sequences are linear functions with slope d.

EXAMPLE 4: Which of the sequences in Example 1 and 2 are arithmetic? What is the common difference?

Ans: Example 3, #2: $c_0 = -3$; $c_{k+1} = c_k + 2$ for $k \geq 1$ is arithmetic: $d = c_{k+1} - c_k = 2$.

GEOMETRIC SEQUENCES: Geometric sequences are exponential functions. To construct a geometric sequence, begin with some initial value, a, and then multiply a fixed amount r to generate successive terms:

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

Recursively we can define a geometric sequence as: $a_1 = a$, $a_{n+1} = r\,a_n$, $n \geq 1$.

The letter 'r' here stands for **common ratio**. To see why, we can solve $a_{n+1} = r\,a_n$ for r: $r = \dfrac{a_{n+1}}{a_n}$.

That is, the ratio r is the same between successive terms.

It doesn't take too much to see that an explicit formula for a_n is: $a_n = ar^{n-1}$ for $n \geq 1$.

This explicit formula shows these sequences are exponential functions with base r.

EXAMPLE 5: Which of the sequences in Example 1 and 2 are geometric? What is the common ratio?

Ans: Example 1, #2: $b_k = \left(\dfrac{2}{3}\right)^k$, $k \geq 0$ is geometric, $r = \dfrac{2}{3}$

Example 2, #1: $b_1 = 2000$, $b_{n+1} = (1.01)b_n$ for $n \geq 1$ is geometric with $r = 1.01$

EXAMPLE 6: (VIDEO) Find an explicit formula for each of the sequences below.

Check your answer by writing out the first few terms.

1. 7, 5, 3, 1, . . . $\qquad a_n = -2n + 9$, $n \geq 1$.

2. 0.9, 0.09, 0.009, 0.0009, . . . $\qquad a_n = (0.9)(0.1)^{n-1}$, $n \geq 1$.

3. 1, -1, 1, -1, . . . $\qquad a_n = (-1)^{n-1}$, $n \geq 1$.

EXAMPLE 7: (VIDEO) Find an explicit formula for each of the sequences below.

Check your answer by writing out the first few terms.

1. $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$ $\qquad a_n = \frac{n}{2n+1},\ n \geq 1.$

2. $2, -\frac{3}{4}, \frac{4}{9}, -\frac{5}{16}, \frac{6}{25} \ldots$ $\qquad a_n = \frac{(-1)^{n-1}(n+1)}{n^2},\ n \geq 1.$

3. $1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$ $\qquad a_n = \frac{1}{n!},\ n \geq 0.$

SHIFTING INDICES: If you write out the following sequences, you'll find them identical:

$$a_n = 4\left(-\frac{1}{3}\right)^{n-1},\ n \geq 1 \qquad\qquad b_k = 4\left(-\frac{1}{3}\right)^{k},\ k \geq 0$$

The difference in the presentation come down to the indices. Indeed, if we set $n-1=k$ so $n=k+1$ we get:

$$4\left(-\frac{1}{3}\right)^{n-1},\ n \geq 1 \quad\longrightarrow\quad 4\left(-\frac{1}{3}\right)^{(k+1)-1},\ (k+1) \geq 1 \quad\longrightarrow\quad 4\left(-\frac{1}{3}\right)^{k},\ k \geq 0$$

In general, if a formula for a sequence a_n is given for $n \geq N$, substituting $n = k+m$ will generate a new formula for the same sequence where the index k ranges from $k \geq N - m$.

EXAMPLE 8: Shift the following sequences to a new index k which begins at $k = 0$.

1. $a_n = 3 - 2n,\ n \geq 1.$

   Ans: $b_k = 1 - 2k,\ k \geq 0.$

2. $a_n = \frac{3n-1}{n+1},\ n \geq 2.$

   Ans: $b_k = \frac{3k+5}{k+3},\ k \geq 0.$

3. $a_n = 4\left(\frac{2}{3}\right)^{2n},\ n \geq 1.$

   Ans: $b_k = 4\left(\frac{2}{3}\right)^{2k+2},\ k \geq 0.$

LIMITS OF SEQUENCES How does Calculus get involved with sequences? As usual, Calculus means limits.

Consider the sequence $a_n = \dfrac{3n}{n+1}$ for $n \geq 0$: $0, \dfrac{3}{2}, 2, \dfrac{9}{4}, \dfrac{12}{5}, \dfrac{5}{2}, \dfrac{18}{7} \ldots$

Is there a particular value L that the numbers a_n tend towards as $n \to \infty$? That is, what is $\lim\limits_{n\to\infty} a_n$?

Recall: $\lim\limits_{x\to\infty} f(x) = L$ means that given $\epsilon > 0$, there is a real number M so that if $x > M$, then $|f(x) - L| < \epsilon$.

The precise definition of $\lim\limits_{n\to\infty} a_n = L$ codifies exactly the the same idea.

DEFINITION: $\lim\limits_{n\to\infty} a_n = L$ means given $\epsilon > 0$ there is a natural number N so that if $n > N$, then $|a_n - L| < \epsilon$.

Paraphrasing, the definition says we can get the terms of the sequence $\{a_n\}$ 'as close as we like to' (within 'ϵ' units of) L by looking 'far enough down the sequence' (for $n > N$).

If the limit of the sequence exists, we say the sequence $\{a_n\}$ **converges** (to L.)

If the limit does not exist, we say the sequence **diverges**.

The similarity between the definitions of the limit of a sequence and the limit of a function gives us the freedom to re-use many of the tools and techniques we've learned in previous chapters. However, there is one power tool, L'Hoptial's Rule, that requires some special attention. Let's return to our original example and consider:

$$\lim_{n\to\infty} \frac{3n}{n+1}$$

One may wonder if it is 'legal' to apply L'Hopital's Rule to this limit. The technical answer is 'no' since L'Hopital's Rule requires **differentiable** functions of a **continuous** variable. However, we can apply L'Hopital's Rule to

$$\lim_{x\to\infty} \frac{3x}{x+1} = \frac{3}{1} = 3$$

Note we can visualize $y = a_n = \dfrac{3n}{n+1}$ as a series of discrete points on the continuous curve $y = f(x) = \dfrac{3x}{x+1}$.

Since all the points on the curve $y = f(x)$ tend towards 3 as $x \to \infty$, it stands to reason that particular points on $y = a_n$ also tend towards 3 as $n \to \infty$. This is in fact true as stated in the following theorem.

THEOREM: If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all $n \geq k$, then $\lim\limits_{n\to\infty} a_n = L$.

Replacing n with x is usually described as 'passing to a continuous variable.' Note that the stipulation '$n \geq k$' in the theorem just means that since we are looking as $n \to \infty$, what happens for any of the 'first' finite terms of the sequence really has no bearing on the limit.

EXAMPLE 9: (VIDEO) Find limits of the following sequences by passing to a continuous variable:

1. $\lim\limits_{n\to\infty} \dfrac{1-n^2}{2n^2-3n+1}$ — Ans: -2

2. $\lim\limits_{n\to\infty} n^2 e^{-2n}$ — Ans: 0

3. $\lim\limits_{n\to\infty} n \ln\left(1 + \dfrac{1}{n}\right)$ — Ans: 1

CAUTION: We have to be care about the logic of the above theorem. IF passing to a continuous variable results in a limit, THEN the corresponding sequences follows suit and converges. However, $\lim_{x\to\infty} \sin(\pi x)$ does not exist yet for the sequence: $\lim_{n\to\infty} \sin(\pi n) = \lim_{n\to\infty} 0 = 0$ since $\sin(\pi n) = 0$ for all natural numbers, n.

Another power tool we have at our disposal from our early limit days is the Squeeze Theorem.

THE SQUEEZE THEOREM: If $b_n \leq a_n \leq c_n$ for all $n \geq k$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} a_n = L$.

EXAMPLE 10: Consider the alternating harmonic sequence: $a_n = \dfrac{(-1)^{n+1}}{n}$, $n \geq 1$.

1. Write out the first few terms of this sequence. Does the sequence appear to have a limit?

   Ans: $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

2. Why does passing to a continuous variable cause problems?

   Ans: $(-1)^{x+1}$ isn't guaranteed to be a real number.

3. Explain why $-\dfrac{1}{n} \leq a_n \leq \dfrac{1}{n}$ for all $n \geq 1$ and use the Squeeze Theorem to prove $\lim_{n\to\infty} a_n = 0$.

Passing to a continuous variable in conjunction with the Squeeze Theorem can be used to prove certain classes of Geometric Sequences converge. We have the following theorem:

LIMITS OF GEOMETRIC SEQUENCES: If $\{a_n\}$ is a geometric sequence with common ratio r:

- If $-1 < r \leq 1$, the sequence converges.
- If $r > 1$ or $r \leq -1$, the sequence diverges.

DEFINITION: A sequence $\{a_n\}$ is called **monotonic** if either the sequence is:

- **non-decreasing**: that is $a_n \leq a_{n+1}$ for all n.
- **non-increasing**: that is $a_n \geq a_{n+1}$ for all n.

EXAMPLE 11: Show the sequence $a_n = \dfrac{3^n}{n!}$ is non-increasing for $n \geq 2$.

To show $\{a_n\}$ is non-increasing, we need to show $a_n \geq a_{n+1}$ if $n \geq 2$.

We start by getting an expression for a_{n+1}: $a_{n+1} = \dfrac{3^{n+1}}{(n+1)!}$.

Next we work to determine when $a_n \geq a_{n+1}$: that is for what values of n is it true that $\dfrac{3^n}{n!} \geq \dfrac{3^{n+1}}{(n+1)!}$.

Rewriting, we get: $\dfrac{(n+1)!}{n!} \geq \dfrac{3^{n+1}}{3^n}$ which simplifies to $n+1 \geq 3$ so $n \geq 2$.

At this point, we note that all of the steps we've taken here are reversible.

Hence, if $n \geq 2$, we have $a_n \geq a_{n+1}$, as required.

DEFINITION: A sequence $\{a_n\}$ is said to be:

- **bounded above** if there is a real number M so that $a_n \leq M$ for all n.
- **bounded below** if there is a real number m so that $a_n \geq m$ for all n.
- **bounded** if the sequence is both bounded above and below.

EXAMPLE 12: Show the sequence $a_n = \dfrac{3^n}{n!}$ is bounded for $n \geq 2$.

We first note that we showed $a_n = \dfrac{3^n}{n!}$ is non-increasing for $n \geq 2$. That means $a_2 \geq a_3 \geq a_4 \ldots$.

Hence, $\{a_n\}$ is bounded above by $a_2 = 4.5$. Moreover, $a_n \geq 0$ for all $n \geq 2$, so $0 \leq a_n \leq 4.5$ for all n.

Bounded monotonic sequences play an important role in describing how the real number system is ?complete? (that is, the real number line has no ?holes? or ?gaps? in it.)

THE (ANALYTIC) COMPLETENESS OF THE REAL NUMBERS: A non-decreasing sequence which is bounded above converges; a non-increasing sequence which is bounded below converges.

EXAMPLE 12: Show the sequence $a_n = \dfrac{3^n}{n!}$, $n \geq 2$, converges.

Since $\{a_n\}$ is non-increasing and bounded below (by 0), we know $\{a_n\}$ converges. But what does it converge to?

Let's call $\lim\limits_{n\to\infty} a_n = L$. Note that $a_{n+1} = \dfrac{3^{n+1}}{(n+1)!} = \dfrac{3}{n+1}\dfrac{3^n}{n!} = \dfrac{3}{n+1}\, a_n$.

Hence, $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{3}{n+1}\, a_n = \lim\limits_{n\to\infty} \dfrac{3}{n+1} \lim\limits_{n\to\infty} a_n = 0 \cdot L = 0$.

But $\lim\limits_{n\to\infty} a_{n+1} = L$ as well (do you see why?) so we get that $L = 0 \cdot L = 0$.

Arguments like the one above can be used to find the limits of recursively defined sequences (once shown to be monotonic and bounded.)

HOMEWORK: Section 10.1: 13 - 55 every other odd, 10.2: 9 - 81 every other odd.

APPENDIX: COMPOUND INTEREST

There are many important applications which require a function to describe interest which is compounded a finite (discrete) number of times per year.[1] Suppose we invest an amount of money, called the **principal**, P, at an annual interest rate r and compound the interest n times per year.

This means the money sits in the account $\frac{1}{n}$ th of a year between compoundings.

Let A_k denote the amount in the account after the k^{th} compounding.

Then A_1 would equal the principal, P, plus the interest accrued over the first $\frac{1}{n}$ th of a year:

$$A_1 = P + Pr\left(\frac{1}{n}\right) = P + P\left(\frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right).$$

During the second compounding, the same process happens only now our starting amount is A_1. Hence, we get:

$$A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2.$$

Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$.

Hence, $\{A_k\}$ is a geometric sequence with common ratio $\left(1 + \frac{r}{n}\right)$.

Since we compound the interest n times per year, after t years, we have nt compounding periods which gives us the formula for compound interest below.

COMPOUND INTEREST: If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount in the account after t years, $A(t)$ is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

It turns out that $\lim\limits_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = Pe^{rt}$, the formula for **continuously** compounded interest.[2]

[1] Simple interest is calculated using: interest = (principal)(rate)(time). Compound interest is interest accrued on interest.

[2] This is a nice exercise using logarithms and L'Hopital's Rule.